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On the Theory of Errors of Physical Geodesy (Gravitational Field of the Earth and Satellite Orbits)

by

Helmut Moritz

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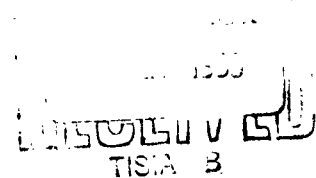The Ohio State University
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FOREWORD

ABSTRACT

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integral transformations are given. They are applied
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at high altitudes from surface gravity anomalies, and
to the accuracy of satellite orbits computed therefrom.

ON THE THEORY OF ERRORS OF PHYSICAL GEODESY

(GRAVITATIONAL FIELD OF THE EARTH AND SATELLITE ORBITS)

1. Introduction

The computations in gravimetric geodesy are mainly evaluations of some integrals; for computing satellite orbits one has to solve differential equations, which again can be reduced to integrations. To these cases, however, the usual theory of error propagation cannot be applied.

We shall make this clear by the example of Stokes' formula. It reads

$$\zeta(\varphi,\lambda) = \frac{R}{4\pi\gamma} \int\limits_{\bar{\varphi}=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_{\bar{\lambda}=0}^{2\pi} S(\varphi,\lambda,\bar{\varphi},\bar{\lambda})\, \Delta g(\bar{\varphi},\bar{\lambda}) \cos\bar{\varphi}\; d\bar{\varphi}\, d\bar{\lambda},$$

where

$$\varphi,\lambda = \text{geographical coordinates,}$$
$$\zeta = \text{geoid undulation,}$$
$$\Delta g = \text{gravity anomaly,}$$
$$R = \text{mean radius of the earth,}$$
$$\gamma = \text{mean theoretical gravity;}$$

$S(\varphi,\lambda,\bar{\varphi},\bar{\lambda})$ is the function of Stokes.

This formula can be considered as a linear integral transformation which correlates the function $\zeta(\varphi,\lambda)$ to the function $\Delta g(\varphi,\lambda)$ --- both gravity anomaly and geoid undulation are functions of position on the sphere.

For deriving the accuracy of ζ from the given accuracy of Δg, we have to investigate the error propagation with respect to linear integral transformations. This case is not dealt with by the usual theory of errors. To be sure, it would be possible to approximate the integral by a sum, but this would be unsatisfactory theoretically and very complicated practically.

In [4] or [5], however, we have given a general theory of errors which can be applied immediately to our case. It is based on the close analogy between functions and vectors on one hand, and integral transformations of functions and linear transformations of vectors on the other hand.

A few results of this theory which are essential for the applications to follow, will now be given and explained, but not derived.

2. <u>General formulas</u>

In order to facilitate understanding, we first consider functions of <u>one</u> variable, instead of functions of <u>two</u> variables φ, λ. A linear integral transformation of such a function $f(x)$ is of form

$$(1) \qquad f^*(x) = \int_{u=\alpha}^{\beta} a(x,u)f(u)du$$

(there correspond $x, u, f, f^*, a(x,u)$ to the quantities (φ, λ), $(\overline{\varphi}, \overline{\lambda})$, Δg, ζ, $S(\varphi, \lambda, \overline{\varphi}, \overline{\lambda})$, respectively, in Stokes' formula).

This linear integral transformation (1) is closely analogous to the ordinary linear transformation,

2

(2) $\qquad f_i^* = \sum_{r=1}^{n} a_{ir} f_r$,

of the quantities f_r which can be considered as a vector. The error propagation for (2) is given by the well-known formula

(3) $\qquad \sigma_{ik}^* = \sum_{r=1}^{n} \sum_{s=1}^{n} a_{ir} a_{ks} \sigma_{rs}$,

where (σ_{ik}) and (σ_{ik}^*) are the error covariance matrices which characterize the accuracy of the f_i's and f_i^*'s, respectively. E.g.,

$$\sigma_{ik} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix} .$$

The diagonal terms are nothing other than the squares of the standard errors m_i of f_i:

(4) $\qquad \sigma_{ii} = m_i^2$,

while σ_{ij}, $i \neq j$, expresses the mutual correlation of f_i and f_j --- if all f_i are independent, all $\sigma_{ij} (i \neq j)$ are zero.

Cf. Grossmann [1], pp. 333 and 334; his formulas (21) and (23) are equivalent to our eqs. (2) and (3), since the matrix of cofactors, Q, is proportional to the error covariance matrix $\sigma : \sigma_{ik} = m_0^2 Q_{ik}$, where m_0 is the standard error of unit weight.

(2) and (1) are completely analogous, there correspond

function $f(x)$ to vector f_i;

(5)   variables x,u to indices i,r;
integral $\int_{u=\alpha}^{\beta}$ to sum $\sum_{r=1}^{n}$.

Hence it is evident that, corresponding to the error covariance matrix which is a quantity with two indices, we shall have a function $\sigma(u,u')$ of two variables u,u', which will be called error covariance function, or error function. Corresponding to (4) we have

(6) $\sigma(u,u) = m^2(u)$,

the square of the standard error $m(u)$ of $f(u)$, while $\sigma(u,u')$ for $u \neq u'$ characterizes the mutual correlation of the values $f(u)$ and $f(u')$.

Since $\sigma(u,u')$, in most cases, is a continuous function, it is even for $u \neq u'$, in general, different from zero, so that there is always correlation. It is, however, often possible to assume $m(u)$ to be independent of u and constant. In this case, $\sigma(u,u')$ is a function of the difference $u-u'$ only. This function often has a shape similar to Fig. 1 only the neighboring values are correlated, the correlation being a decreasing function of distance $|u-u'|$.
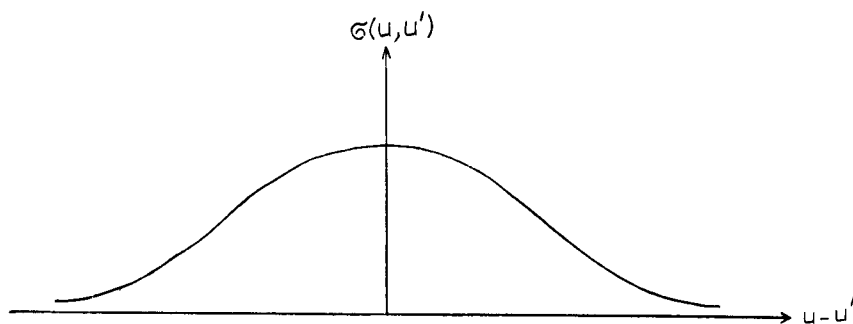
Fig. 1

Functions similar to Fig. 1 are, e.g.,

(7a) $\quad \sigma(u,u') = \sigma_0 e^{-c^2(u-u')^2}$

or

(7b) $\quad \sigma(u,u') = \dfrac{\sigma_0}{1+k^2(u-u')^2}$.

From (3), by means of the analogies (5), we find the formula of error propagation in linear integral transformations (1):

(8) $\quad \sigma*(x,x') = \displaystyle\int_{u=\alpha}^{\beta} \int_{u'=\alpha}^{\beta} a(x,u)\, a(x',u')\, \sigma(u,u')\, du\, du'$.

We get an important special case if we need the function f*(x) in one fixed point $x=x_0$ only. Here we introduce the following notations

$$f*(x_0) \equiv F, \quad \sigma*(x_0,x_0) = m*^2(x_0) \equiv M^2, \quad a(x_0,u) \equiv h(u) .$$

F is a <u>linear functional</u> of f(x); according to (1) it is given by

(9) $\quad F = \displaystyle\int_{u=\alpha}^{\beta} h(u)\, f(u)\, du$.

The corresponding formula of error propagation is

(10) $\quad M^2 = \displaystyle\int_{u=\alpha}^{\beta} \int_{u'=\alpha}^{\beta} h(u)\, h(u')\, \sigma(u,u')\, du\, du'$.

Using (5), we have in the analogous case of the vector f_i.

(11) $\quad F = \sum\limits_{r=1}^{n} h_r f_r$

and

(12) $\quad M^2 = \sum\limits_{r=1}^{n} \sum\limits_{s=1}^{n} h_r h_s \sigma_{rs}$,

i.e., the well-known elementary case of a linear function of the variables $f_1, \ldots f_n$ and its error propagation. So our linear functional simply corresponds to a linear function.

Between (1) and (2) and between (9) and (11) we have further analogies which are still closer. If we develop a function $f(x)$ in a series with respect to a system of orthogonal functions, the coefficients of this development form an infinite vector; and if we perform the same development for the error function $\sigma(x,x')$, the coefficients form an "infinite matrix." With these infinite quantities we can operate in exactly the same way as with finite vectors and matrices. Hence, eqs. (2), (3), (11), and (12) hold immediately for this case; we only have to replace, in the summation sign, n by ∞. Thus, any linear integral transformation is equivalent to a linear transformation of the coefficients of an orthogonal expansion.

Such orthogonal expansions are, e.g., Fourier series or expansions in spherical harmonics. They are important because a very simple transformation of the coefficients may correspond to a complicated integral transformation. A good example is Stokes' formula; this fact is used in deriving this formula by means of spherical harmonics.

Since spherical harmonics are of fundamental importance in physical geodesy, the error theory of orthogonal functions is important, too. In the examples treated in this paper we do not need them, however. Therefore we cannot present them here. The reader will find a comprehensive treatment of the whole subject in [5].

3. Accuracy of the Gravity Vector at High Altitudes

Let us now apply these general formulas to problems of physical geodesy. We assume gravity on the earth's surface given with a certain accuracy. From these data we compute the gravity vector outside the earth. By means of this external gravity vector we can calculate

satellite orbits. <u>How accurate is the computation of the external</u>

<u>gravity vector and of satellite orbits?</u> We shall simplify our assumptions, in order to make clear the structure of the problem.

Besides the actual gravity vector $\vec{y}$, we introduce the theoretical gravity vector $\vec{\gamma}$, referred to the same point, and the vector of gravity disturbance, $\vec{\delta}$, so that

$$\vec{y} = \vec{\gamma} + \vec{\delta}$$

and

(13) $\vec{\delta} = -\text{grad } T$.

Hence, $\vec{\delta}$ is the gradient vector of the disturbing potential T which is defined as the difference of actual gravity potential W and theoretical gravity potential U.

T may be expressed, according to Helmert, by a "coating" with density

(14) $\mu = \Delta g + \dfrac{3\gamma}{2R} \zeta,$

namely,

(15) $T = \dfrac{1}{2\pi} \iint \dfrac{\mu}{D} \, dS.$

The integration is performed over the globe; D is the distance of the surface element dS from the point P in which T is computed.

In our case it is sufficient to replace the globe by its tangential plane, so that by (15),

$$(15') \quad T(x,y) = \frac{1}{2\pi} \int_{\xi=-\infty}^{\infty} \int_{\eta=-\infty}^{\infty} \frac{\mu(\xi,\eta)}{D} \, d\xi d\eta \; .$$

Here is

$$(16) \quad D^2 = (x-\xi)^2 + (y-\eta)^2 + H^2 \; ;$$

x,y are plane rectangular coordinates of point P ; H is the height above ground; and ξ, η are the coordinates of dS.

Denoting the components of the vector of gravity disturbance by δ_x, δ_y, δ_z, we find by (13),

$$\delta_x = -\frac{\partial T}{\partial x}, \quad \delta_y = -\frac{\partial T}{\partial y}, \quad \delta_z = -\frac{\partial T}{\partial H} \; .$$

Differentiating $(15')$ with respect to x, y, H yields

$$\delta_x(x,y) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{\mu(\xi,\eta)}{D^3} (x-\xi) \, d\xi \, d\eta \, ,$$

$$(17) \qquad \delta_y(x,y) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{\mu(\xi,\eta)}{D^3} (y-\eta) \, d\xi \, d\eta \, ,$$

$$\delta_z(x,y) = \frac{H}{2\pi} \iint_{-\infty}^{\infty} \frac{\mu(\xi,\eta)}{D^3} \, d\xi \, d\eta \; .$$

These formulas are integral transformations which correlate the functions δ_x, δ_y, δ_z to the function μ.

For applying the general formulas of the foregoing section to this

8

problem, we have to consider that here we have functions of <u>two</u> variables, the error functions of which are, therefore, functions of <u>four</u> variables. The coordinate pairs (x,y) and (ξ,η) correspond to the single variables x and u, in the previous section; instead of a single integral we now have a double integral. Denoting the error functions of $\delta_x(x,y)$, $\delta_y(x,y)$, $\delta_z(x,y)$ and of $\mu(x,y)$ by $\sigma_x(x,y,x',y')$, $\sigma_y(x,y,x',y')$, $\sigma_z(x,y,x',y')$, and $\sigma(x,y,x',y')$, respectively, we find by (8) immediately

$$\sigma_x(x,y,x',y') = \frac{1}{4\pi^2} \int\!\!\int\!\!\int\!\!\int_{-\infty}^{\infty} \frac{\sigma(\xi,\eta,\xi',\eta')}{D^3 D'^3} (x-\xi)(x'-\xi')\,d\xi\,d\eta\,d\xi'\,d\eta',$$

$$(18) \qquad \sigma_y(x,y,x',y') = \frac{1}{4\pi^2} \int\!\!\int\!\!\int\!\!\int_{-\infty}^{\infty} \frac{\sigma(\xi,\eta,\xi',\eta')}{D^3 D'^3} (y-\eta)(y'-\eta')\,d\xi\,d\eta\,d\xi'\,d\eta',$$

$$\sigma_z(x,y,x',y') = \frac{H^2}{4\pi^2} \int\!\!\int\!\!\int\!\!\int_{-\infty}^{\infty} \frac{\sigma(\xi,\eta,\xi',\eta')}{D^3 D'^3}\,d\xi\,d\eta\,d\xi'\,d\eta',$$

where D is given by (16) and D' by

$$D'^2 = (x'-\xi')^2 + (y'-\eta')^2 + H^2 .$$

For simplicity, we assume the error function σ to have the form (7a) (two-dimensional):

$$(19) \qquad \sigma(\xi,\eta,\xi',\eta') = \sigma_0\, e^{-c^2[(\xi-\xi')^2 + (\eta-\eta')^2]} = \sigma_0\, e^{-c^2 d^2},$$

9

i.e., the correlation depends only on the distance

$$d = \sqrt{(\xi-\xi')^2 + (\eta-\eta')^2}$$

of points (ξ,η) and (ξ',η'). (19) being appreciably different from zero for $(\xi,\eta) \doteq (\xi',\eta')$ only, we may set in (18), approximately,

$$\xi' = \xi, \quad \eta' = \eta .$$

So we get, say, for σ_x

$$\sigma_x(x,y,x',y') = \frac{1}{4\pi^2} \iiiint\limits_{-\infty}^{\infty} \frac{\sigma_0 e^{-c^2 d^2}}{D^3 D''^3} (x-\xi)(x'-\xi)\, d\xi\, d\eta\, d\xi' d\eta',$$

where

$$D''^2 = (x'-\xi)^2 + (y'-\eta)^2 + H^2 .$$

Now we first integrate with respect to ξ', η', where $x-\xi$, $x'-\xi$, D and D" are constant for the integration. Since

$$\int\limits_{\xi'=-\infty}^{\infty} \int\limits_{\eta'=-\infty}^{\infty} \sigma_0 e^{-c^2\left[(\xi-\xi')^2+(\eta-\eta')^2\right]}\, d\xi' d\eta' =$$

$$= \sigma_0 \cdot \int\limits_{-\infty}^{\infty} e^{-c^2(\xi-\xi')^2} d\xi' \cdot \int\limits_{-\infty}^{\infty} e^{-c^2(\eta-\eta')^2} d\eta' = \sigma_0 \cdot \frac{\sqrt{\pi}}{c} \cdot \frac{\sqrt{\pi}}{c} =$$

$$= \pi \frac{\sigma_0}{c^2} ,$$

we obtain

$$(20) \quad \sigma_x(x,y,x',y') = \frac{1}{4\pi} \frac{\sigma_0}{c^2} \iint\limits_{-\infty}^{\infty} \frac{(x-\xi)(x'-\xi)}{D^3 D''^3}\, d\xi\, d\eta .$$

Even now direct integration is very difficult, resulting in elliptic integrals.

We may, however, avail ourselves of a development in a series. By

$$\xi - x = r \cos \varphi, \quad \eta - y = r \sin \varphi$$

and

(21) $\quad x' - x = s \cos \alpha, \quad y' - y = s \sin \alpha$

we introduce polar coordinates, the origin being point $P(x,y)$ (Fig. 2).
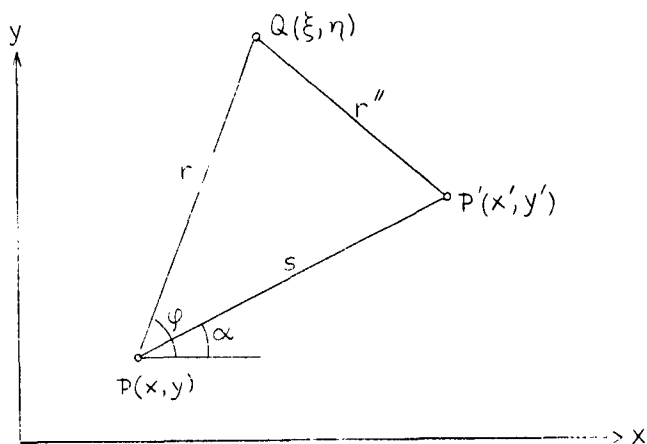
Fig. 2

Then,

$$D^2 = H^2 + r^2, \quad D''^2 = H^2 + r''^2 = H^2 + r^2 - 2rs \cos(\varphi - \alpha) + s^2 \ .$$

Developing, for small s, in a power series with respect to s, we get

$$\frac{1}{D''^3} = \frac{1}{D^3} + \frac{3r \cos(\varphi - \alpha)}{D^5} s + \left(- \frac{3}{2 D^5} + \frac{15}{2} \frac{r^2 \cos^2(\varphi - \alpha)}{D^7} \right) s^2 + \ldots$$

11

Inserting this in (20) and considering

$$x-\xi = -r \cos \varphi, \qquad x'-\xi = -r \cos \varphi + s \cos \alpha ,$$

$$\frac{1}{D^3} = \frac{1}{(r^2+H^2)^{3/2}} , \qquad d\xi d\eta = r \, dr \, d\varphi ,$$

we may perform the integration in polar coordinates relatively easily. In a similar way we treat σ_y and σ_z, finally obtaining

$$\sigma_x(x,y,x',y') = \frac{\sigma_0}{16c^2H^2}\left[1 - \frac{3}{16}\frac{s^2}{H^2}(1+2\cos^2\alpha) + \ldots\right] ,$$

$$(22) \qquad \sigma_y(x,y,x',y') = \frac{\sigma_0}{16c^2H^2}\left[1 - \frac{3}{16}\frac{s^2}{H^2}(1+2\sin^2\alpha) + \ldots\right] ,$$

$$\sigma_z(x,y,x',y') = \frac{\sigma_0}{8c^2H^2}\left[1 - \frac{3}{8}\frac{s^2}{H^2} + \ldots\right] .$$

The polar coordinates s, α are related to x, y, x', y' by (21). We see that the expressions (22) depend on the mutual position of points P and P' only: σ_z depends only on the distance s, σ_x and σ_y on the direction α, too.

According to (6) we get the mean square errors of δ_x, δ_y, δ_z by setting in (22), $x' = x$, $y' = y$, i.e., $s=0$. Hence,

$$(23) \qquad m_x^2 = m_y^2 = \frac{\sigma_0}{16c^2H^2}, \qquad m_z^2 = \frac{\sigma_0}{8c^2H^2} .$$

$m_x = m_y$ and m_z have the same order of magnitude; they are inversely

12

proportional to the elevation H above ground. These formulas become invalid for small H because, in this case, (23) tends to infinity. For small H, indeed, the simplifications leading to (20) would not have been admissible.

More detailed investigations show that the error functions (22) can, in sufficient approximation, be represented by functions

$$\frac{m^2}{1+k^2 s^2} = m^2(1-k^2 s^2+\ldots).$$

Thus we find from (22),

$$\sigma_x(s,\alpha) = \frac{m_x^2}{1+\frac{3}{16}\frac{s^2}{H^2}\left(1+2\cos^2\alpha\right)} \quad,$$

$$(24) \quad \sigma_y(s,\alpha) = \frac{m_y^2}{1+\frac{3}{16}\frac{s^2}{H^2}\left(1+2\sin^2\alpha\right)} \quad,$$

$$\sigma_z(s) = \frac{m_z^2}{1+\frac{3}{8}\frac{s^2}{H^2}} \quad.$$

These are the desired error functions of δ_x, δ_y, δ_z, or of the components g_x, g_y, g_z of the gravity vector $\vec{y}$.

4. Influence on Satellite Orbits

In the last section we have evaluated the accuracy of the gravity vector (24); now we are going to estimate the influence of this accuracy on computed satellite orbits.

Essential for the following are the formulas for the change of the orbital parameters a, e, i, Ω, caused by a disturbing force (X, Y, Z), which are found in any textbook on celestial mechanics (cf. [8]):

$$\frac{da}{dt} = 2a \; \varkappa[X \sec \varphi(1 + e \cos v) + Z \; \text{tg} \; \varphi \sin v],$$

$$\frac{de}{dt} = \varkappa \cos \varphi[X(\cos v + \cos E) + Z \sin v],$$

$$\frac{di}{dt} = \frac{r}{a} \frac{\varkappa}{\cos \varphi} \; Y \cos u,$$

$$\frac{d\Omega}{dt} = \frac{r}{a} \frac{\varkappa}{\cos \varphi} \; Y \frac{\sin u}{\sin i} \; .$$

Here we have used the following notations:

    a = semi-major axis of the satellite orbit,
    e = numerical excentricity of the orbit,
    i = inclination of the orbit,
    Ω = right ascension of the ascending node,
    t = time
    r = radius vector
    v = true anomaly
    E = excentric anomaly
    u = longitude in the orbit, reckoned from the node (cf. [3]).
φ is an auxiliary angle connected with the excentricity e by the relation
    e = sin φ .
$\varkappa$ is a constant defined by

$$\varkappa = \sqrt{\frac{a}{fM}}$$

(f = Newton's constant of gravitation, M = mass of the earth). X, Y, Z are the components of the disturbing force, namely

   Z directed along the radius vector,
   Y normal to the orbital plane,
   X normal to Y and Z.

We restrict our problem to the simplest and practically important case of an approximately circular orbit. Here,

$$e = \sin \varphi \doteq 0, \quad r \doteq a, \quad E \doteq v,$$

and we have

$$\frac{da}{dt} = 2a \, \varkappa \, X,$$

$$\frac{de}{dt} = 2 \, \varkappa \, X \cos v + \varkappa \, Z \sin v,$$

(25)

$$\frac{di}{dt} = \varkappa \, Y \cos u,$$

$$\frac{d\Omega}{dt} = \varkappa \, Y \, \frac{\sin u}{\sin i} \, .$$

X is now directed along the tangent to the orbit; X and Y are horizontal, Z is vertical. Hence, we can identify the directions of the coordinate axes x, y, z of the foregoing section, with the directions of X, Y, Z. Then, the components X, Y, Z of the disturbing force are nothing other than the gravity disturbances δ_x, δ_y, δ_z.

We consider first da/dt. Denoting by Δa the finite variation of a for a whole revolution (period of revolution = T), we obtain

$$\Delta a = \int_{t=0}^{T} \frac{da}{dt} \, dt = \int_{u=0}^{2\pi} \frac{da}{dt} \, \frac{dt}{du} \, du = \frac{T}{2\pi} \int_{u=0}^{2\pi} \frac{da}{dt} \, du,$$

15

hence

$$\Delta a = 2a \, \varkappa \, \frac{T}{2\pi} \int_0^{2\pi} X(u) \, du \; .$$

This is a linear functional of form (9)($f(u) = X(u)$, $h(u) = 2a \, \varkappa \, T/2\pi = $ const.). Eq. (10) yields immediately

$$(26a) \quad M_a^2 = 4a^2 \varkappa^2 \, \frac{T^2}{4\pi^2} \int_{u=0}^{2\pi} \int_{u'=0}^{2\pi} \sigma_x(u,u') \, du \, du' \; .$$

Analogously,

$$M_e^2 = 4\varkappa^2 \, \frac{T^2}{4\pi^2} \int_{v=0}^{2\pi} \int_{v'=0}^{2\pi} \sigma_x(v,v') \cos v \cos v' \, dv \, dv' \; +$$

$$(26b)$$

$$+ \, \varkappa^2 \, \frac{T^2}{4\pi^2} \int_{v=0}^{2\pi} \int_{v'=0}^{2\pi} \sigma_z(v,v') \sin v \sin v' \, dv \, dv'$$

(since δ_x and δ_y can be shown to be practically uncorrelated), and

$$(26c) \quad M_i^2 = \varkappa^2 \, \frac{T^2}{4\pi^2} \int_{u=0}^{2\pi} \int_{u'=0}^{2\pi} \sigma_y(u,u') \cos u \cos u' \, du \, du' \; ,$$

$$(26d) \quad M_\Omega^2 = \frac{\varkappa^2}{\sin^2 i} \, \frac{T^2}{4\pi^2} \int_{u=0}^{2\pi} \int_{u'=0}^{2\pi} \sigma_y(u,u') \sin u \sin u' \, du \, du' \; .$$

The error functions σ_x, σ_y, σ_z are given by (24). The x-axis being tangent to the orbit, we have along the orbit

16

$$\alpha = 0, \quad s = a(u-u') = a(v-v')$$

(u and v are angles). Hence,

$$\sigma_x(u,u') = \frac{\sigma_0}{16c^2H^2} \left[1 + \frac{9a^2}{16H^2}(u-u')^2\right]^{-1},$$

$$\sigma_y(u,u') = \frac{\sigma_0}{16c^2H^2} \left[1 + \frac{3a^2}{16H^2}(u-u')^2\right]^{-1},$$

$$\sigma_z(u,u') = \frac{\sigma_0}{8c^2H^2} \left[1 + \frac{3a^2}{8H^2}(u-u')^2\right]^{-1}.$$

Inserting this in eqs. (26) and applying a similar trick as in integrating (18), we find in sufficient approximation finally

(27)
$$M_a^2 = \frac{1}{6}a^2Q^2, \qquad M_e^2 = \frac{4+\sqrt{6}}{48}Q^2,$$

$$M_i^2 = \frac{\sqrt{3}}{48}Q^2, \qquad M_\Omega^2 = \frac{1}{\sin^2 i}\frac{\sqrt{3}}{48}Q^2,$$

where

$$Q^2 = \frac{T^2}{fM}\frac{\sigma_0}{c^2}\frac{1}{H}$$

is a nondimensional quantity.

By (27) we have immediately

(27')
$$M_a = 0.408\,aQ, \qquad M_e = 0.367\,Q,$$
$$M_i = 0.190\,Q, \qquad M_\Omega = 0.190\,Q/\sin i.$$

These are the mean square errors of the variations Δa, Δe, Δi, $\Delta\Omega$ for one revolution, which are caused by the inaccuracies of the given gravity values, measured on the earth's surface. They are valid, provided

that the error function of the coating μ can be represented with sufficient accuracy by a function (19) with parameters σ_o, c.

In order to get the mean square errors of the variations of orbital elements for _several_ revolutions, we obviously have only to multiply the values (27') by the square root of the number of revolutions.

5. A Numerical Example

We have computed some numerical values for an elevation

$$H = 500km$$

above ground. For σ_o and c we choose some characteristic pairs of values. The results are shown in Table 1.

Table 1

σ_o mgal	c^{-1} km	$m_x = m_y$ mgal	m_z mgal	M_a m	M_e	$M_i = M_\Omega \sin i$ seconds of arc
2	15	0.01	0.02	0.3	0.04×10^{-6}	0.005
10	25	0.1	0.2	3	0.4×10^{-6}	0.04
15	45	0.3	0.5	7	1.0×10^{-6}	0.11
19	90	0.9	1.2	18	2.5×10^{-6}	0.27
20	220	2.2	3.1	46	6.6×10^{-6}	0.69

The pairs $\sqrt{\sigma_o}$, c^{-1} have the following meaning. We make the simplifying, but not quite correct, assumption that the inaccuracies in μ(14) are caused only by Δg, but not by ζ. Then, interpolation being the main source of error in Δg, σ is the error function for interpolation of gravity, which is, in fact, of form (19). $\sqrt{\sigma_o}$ is the mean square error of integration, and c has the dimension of a reciprocal length, so that $\sqrt{\sigma_o}$ and c^{-1} are more intuitive than σ_o and c. The pairs of values

given may correspond to the interpolation error for average station distances of about 10, 50, 100, 200 and 500 km [7].*

The values found in this way are very small. Even if we correctly have regard for the errors of the geoid undulations, its order of magnitude will not be changed, as can be seen on closer examination. A necessary condition is, however, uniform coverage of the whole earth by gravity stations.

We have deliberately simplified conditions as much as possible because of the difficulty of the problem. More realistic estimations will have to take into account, e.g., the errors of ζ and, above all, the big gaps in the world gravity net which considerably impair the accuracy.

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* These values are consistent also with the covariance function given by Kaula (J. Geophys. Res. 64, 2401-2421, 1959).

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